

LOCALIZATION STUDY OF A NON-LOCAL ENERGETIC DAMAGE MODEL

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ABSTRACT. The contribution presents an analysis of a rate-independent non-local damage model, recently proposed by Mielke and Roubíček (2006). An analytical as well as numerical solution of a simple one-dimensional bifurcation problem is performed, demonstrating that, for the elementary localization test, the model is free of pathological features.

Keywords. damage, non-locality, energetic solution, discretization in time, bifurcation

1. INTRODUCTION

In engineering mechanics, *damage* is understood as a load-induced evolution of microstructural defects, resulting in a reduced macroscopic material integrity. The phenomenological constitutive models of damage incorporate the irreversible phenomena by reducing the secant modulus of elasticity depending on the *internal* damage variable. Since the seminal contribution of Bažant (1976), it has been well-understood that such description within the framework of local (i.e. scale-free) continuum mechanics leads to an ill-posed problem, resulting in localization of damage growth into an arbitrarily small region. As a remedy to this pathology, a plethora of non-local rate-independent continuum theories, based on integral, explicit and implicit gradient approaches, has been proposed to introduce an *internal length scale* into the description, see e.g. (Jirásek and Bažant, 2002, Chapter 26) for a representative overview.

Despite a significant increase in objectivity offered by the enhanced continuum theories, the non-local damage formulations often suffer from the fact that the non-local variables are introduced into the model in an ad-hoc fashion, thus violating basic constraints of thermodynamics. In addition, due to violation of the principle of local action, such inconsistencies are rather difficult to detect, especially in the multi-dimensional setting, e.g. (Peerlings et al., 2004). Fortunately, as first demonstrated by Jirásek (1998) and later confirmed by a number of independent studies, e.g. (Peerlings et al., 2001; Jirásek and Rolshoven, 2003; Di Luzio and Bažant, 2005; Engelen et al., 2006), a simple one-dimensional study of the localization behavior can serve as a convenient “filter” test, allowing to pinpoint various inconsistencies in the constitutive model formulation.

Motivation of the present work arose from an energetic non-local damage model proposed by Mielke and Roubíček (2006), which combines the basic features of an engineering damage model due to Frémond and Nedjar (1996) with recent advances in the mathematical theory of rate-independent irreversible processes (Mielke, 2005). Such a connection

Dedicated to Zdeněk Bittnar on the occasion of his 65 birthday.

provides a powerful modeling and analysis framework, allowing at the same time a rigorous mathematical treatment of the complete damage (Bouchitté et al., 2007), theoretically supported numerical implementation (Mielke et al., 2007) as well as a thermodynamically consistent variational formulation of the non-local damage evolution problem. In this contribution, a localization study of the model will be performed to examine its qualitative behavior from the engineering viewpoint. Numerical as well as analytical results are provided to illustrate its basic features with emphasis on the proper representation of the damage process from its initiation until complete failure.

2. GLOBAL ENERGETIC FORMULATION

Let us consider a prismatic bar of length L , subjected to displacement-controlled uniaxial tensile loading. In the sequel, the bar will be represented by the interval $\Omega = (-L/2; L/2)$, with boundary Γ (consisting of two points) subjected to the Dirichlet loading $u_D(t)$ (see Figure 1), where t denotes the (pseudo-) time taken from interval $\langle 0; T \rangle$.

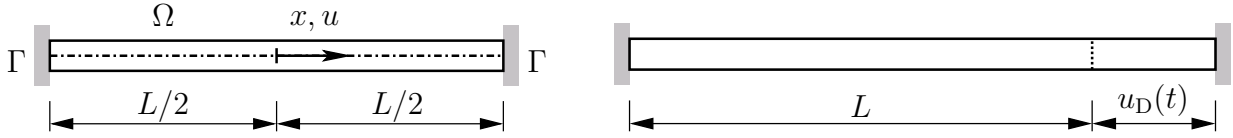


FIGURE 1. Bar under uniaxial displacement-controlled tension.

Following the standard thermodynamic approach to constitutive modeling, cf. (Jirásek and Bažant, 2002, Chapter 25), a state of the system is described using an admissible displacement and damages fields \hat{u} and $\hat{\omega}$, respectively. Formally, we write

$$\hat{u} \in \mathbb{K}(t) = \left\{ \hat{u} \in W_2^{(2)}(\Omega), \hat{u}(x)|_{\Gamma} = u_D(t) \right\} \quad (1)$$

$$\hat{\omega} \in \mathbb{Z} = \left\{ \hat{\omega} \in W_2^{(2)}(\Omega), 0 \leq \hat{\omega}(x) \leq 1 \text{ a.e. in } \Omega \right\} \quad (2)$$

where \mathbb{K} denotes the set of kinematically admissible displacements, \mathbb{Z} stands for the set of admissible damage values, and $W_2^{(2)}(\Omega)$ is the Sobolev space of functions with square-integrable weak derivatives, e.g. (Rektorys, 1982), often denoted as $H^1(\Omega)$.

Within the adopted *global* energetic framework (Mielke, 2005), the constitutive description of the damage model is provided by the *globally stored energy* functional

$$\mathcal{E}(t, \hat{u}, \hat{\omega}) = \frac{1}{2} \int_{\Omega} (1 - \hat{\omega}(x)) E (\hat{u}'(x))^2 dx + \frac{1}{2} \int_{\Omega} g_f \ell^2 (\hat{\omega}'(x))^2 dx \quad (3)$$

and by the *global dissipation distance*

$$\mathcal{D}(\hat{\omega}^1, \hat{\omega}^2) = \begin{cases} \int_{\Omega} g_f (\hat{\omega}^2(x) - \hat{\omega}^1(x)) dx & \text{if } \hat{\omega}^2(x) \geq \hat{\omega}^1(x) \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases} \quad (4)$$

Physically, \mathcal{E} represents the energy reversibly stored in the system, and \mathcal{D} is the energy dissipated by changing the damage field from $\hat{\omega}^1$ to $\hat{\omega}^2$. In Eqs. (3) and (4), E [Pa] denotes

the Young modulus of the material, g_f [Jm^{-3}] is the amount of energy needed to disintegrate a unit volume of the material, and ℓ [m] is a characteristic length of the model. Later it will become clear that the “ $+\infty$ ” term appearing in Eq. (4) ensures the irreversibility of the damage evolution; i.e. at any point, the damage variable cannot decrease in time.

Now, given the Dirichlet loading $u_D(t)$, energetic functionals \mathcal{E} and \mathcal{D} and initial data $u(0)$ and $\omega(0)$, the *energetic solution* of the damage problem is provided by functions $u(t)$ and $\omega(t)$ satisfying (Mielke, 2005):

Global stability: for all $t \in (0; T]$, $\hat{u} \in \mathbb{K}(t)$ and $\hat{\omega} \in \mathbb{Z}$

$$\mathcal{E}(t, u(t), \omega(t)) \leq \mathcal{E}(t, \hat{u}(t), \hat{\omega}(t)) + \mathcal{D}(\omega(t), \hat{\omega}) \quad (5)$$

Energy equality: for all $t \in (0; T]$

$$\mathcal{E}(t, u(t), \omega(t)) + \mathcal{D}(\omega(0), \omega(t)) = \mathcal{E}(0, u(0), \omega(0)) + \int_0^t \mathcal{P}(s, u(s), \omega(s)) \, ds \quad (6)$$

where \mathcal{P} denotes the power of external loading.

Although the previous two conditions present the formal definition of the energetic solution, the analysis itself will be performed using the time discretization technique, see e.g. (Rektorys, 1982) for a nice exposition. To that end, we introduce a uniform partitioning of the time interval $0 = t_0 < t_1 = t_0 + \Delta t < \dots < t_N = T$ and inductively solve the minimization problem

$$(u(t_k), \omega(t_k)) \in \text{Arg} \min_{(\hat{u}, \hat{\omega}) \in \mathbb{K}(t_k) \times \mathbb{Z}} [\mathcal{E}(t_k, \hat{u}, \hat{\omega}) + \mathcal{D}(\omega(t_{k-1}), \hat{\omega})] \quad \text{for } k = 1, 2, \dots, N \quad (7)$$

Note that the previous problem is independent from Δt , which is consistent with the assumed rate-independent character of the damage process. The theoretical results gathered in (Mielke and Roubíček, 2006; Bouchitté et al., 2007; Mielke et al., 2007) show that, under reasonable data qualification, the solution of the time-discretized problem converge to the energetic solution as $\Delta t \rightarrow 0$.

3. ANALYSIS OF LOCALIZATION BEHAVIOR

Having established the essentials of the global energetic framework, we will now proceed with the main goal of this contribution – the analysis of the simple uniaxial tensile test. Intuitively, we expect that as long as the energy density at all points of the structure stays below g_f , the response of the bar remains elastic and no damage evolution occurs within the structure. After the energy density reaches the value of g_f at a certain time t_k , damage is initiated and localization occurs. Therefore, the explicit description of the localization phenomenon will build on the incremental variational principle (7). Note that, for the sake of brevity, the time instant t_k and the time increment Δt will be omitted in the rest of this paper.

Dropping the constraints on ω for a moment, the optimality conditions for the minimization problem (7) read

$$\int_{\Omega} \left[\delta u'(x)(1 - \omega(x))E u'(x) - \frac{1}{2} \delta \omega(x) E (u'(x))^2 + \delta \omega'(x) g_f \ell^2 \omega'(x) + \delta \omega(x) g_f \right] \, dx \geq 0$$

for all admissible variations δu vanishing on Γ and arbitrary $\delta\omega$. Assuming the involved fields are smooth enough, we formally obtain

$$- \int_{\Omega} \delta u(x) \left((1 - \omega(x)) E u'(x) \right)' dx - \int_{\Omega} \delta \omega(x) \left(\frac{1}{2} E (u'(x))^2 + g_f \ell^2 \omega''(x) - g_f \right) dx + \left[\delta \omega(x) g_f \ell^2 \omega'(x) \right]_{\Gamma} \geq 0$$

Using the arbitrariness of the variations and imposing the constraints $0 \leq \omega \leq 1$ finally leads to the system of the Karush-Kuhn-Tucker conditions, cf. (Jirásek and Bažant, 2002, Section 15.2.4)

$$\left((1 - \omega(x)) E u'(x) \right)' = 0 \quad (8)$$

$$g_f \ell^2 \omega''(x) + \frac{1}{2} E (u'(x))^2 \leq g_f \text{ for } \omega(x) = 0 \quad (9)$$

$$g_f \ell^2 \omega''(x) + \frac{1}{2} E (u'(x))^2 = g_f \text{ for } 0 < \omega(x) < 1 \quad (10)$$

$$g_f \ell^2 \omega''(x) + \frac{1}{2} E (u'(x))^2 \geq g_f \text{ for } \omega(x) = 1 \quad (11)$$

Eq. (8) corresponds to the uniform stress condition

$$(1 - \omega(x)) E u'(x) = \text{const} = \sigma \quad (12)$$

while Eqs. (9), (10) and (11) provide the link between strain and damage and describe the elastic, damaging and fully damaged part of the structure, respectively. Note that by virtue of Eq. (12), the stress σ remains to be well-defined even in the fully damaged region, where the physical meaning of displacements u and strains u' is rather questionable. This opens the way to the rigorous mathematical treatment of the complete damage problem, see (Bouchitté et al., 2007; Mielke et al., 2007) for further discussion.

Concentrating on the damaging zone only, we divide Eq. (10) by g_f and, employing identity (12), rewrite it in the form of a non-linear ordinary differential equation

$$\ell^2 \omega''(x) + \frac{\mu}{(1 - \omega(x))^2} = 1 \quad (13)$$

where

$$\mu = \frac{\sigma^2}{2 E g_f} \quad (14)$$

parameterizes the softening process and ranges from $\mu = 1$ at the onset of localization to $\mu = 0$ at the stress-free state of complete failure. It is also useful to rewrite Eq. (13) in the rate form

$$\ell^2 \dot{\omega}''(x) + 2 \frac{\mu}{(1 - \omega(x))^3} \dot{\omega}(x) = - \frac{\dot{\mu}}{(1 - \omega(x))^2} \quad (15)$$

which is an ordinary second-order differential equation for the damage rate $\dot{\omega}$ with possibly non-constant coefficients. Note that damage irreversibility requires $\dot{\omega} > 0$.

3.1. Bifurcation from uniform state at damage initiation. Observe that at the onset of damage, $\omega(x) = 0$ and $\mu = 1$. Therefore, Eq. (15) simplifies to

$$\ell^2 \dot{\omega}''(x) + 2\dot{\omega}(x) = -\dot{\mu} \quad (16)$$

leading to the general solution

$$\dot{\omega}(x) = -\frac{1}{2}\dot{\mu} + C_1 \cos \frac{\sqrt{2}(x - x_s)}{\ell} \quad (17)$$

with C_1 and x_s denoting the integration constants, the latter specifying the position of the center of the localized profile. Assuming that this zone corresponds to an interval (x_a, x_b) located sufficiently far from the boundary, the damage rate has to verify four independent conditions

$$\dot{\omega}(x_a) = \dot{\omega}(x_b) = 0 \quad \dot{\omega}'(x_a) = \dot{\omega}'(x_b) = 0 \quad (18)$$

yielding

$$x_s = \frac{1}{2}(x_a + x_b), \quad x_b - x_a = L_1 = \sqrt{2}\pi\ell, \quad C_1 = -\frac{1}{2}\dot{\mu} \quad (19)$$

where L_1 is the width of the localized zone, cf. Figure 2a. The rate of the damage variable at the onset of damage is therefore given by

$$\dot{\omega}(x) = -\frac{1}{2}\dot{\mu} \left(1 + \cos \frac{\sqrt{2}(x - x_s)}{\ell} \right), \quad x \in \left(x_s - \frac{\pi\ell}{\sqrt{2}}; x_s + \frac{\pi\ell}{\sqrt{2}} \right) \quad (20)$$

The previous expression is consistent with the constraint $\dot{\omega} \geq 0$ since the stress rate $\dot{\sigma}$ is negative in the softening regime, which implies $\dot{\mu} < 0$.

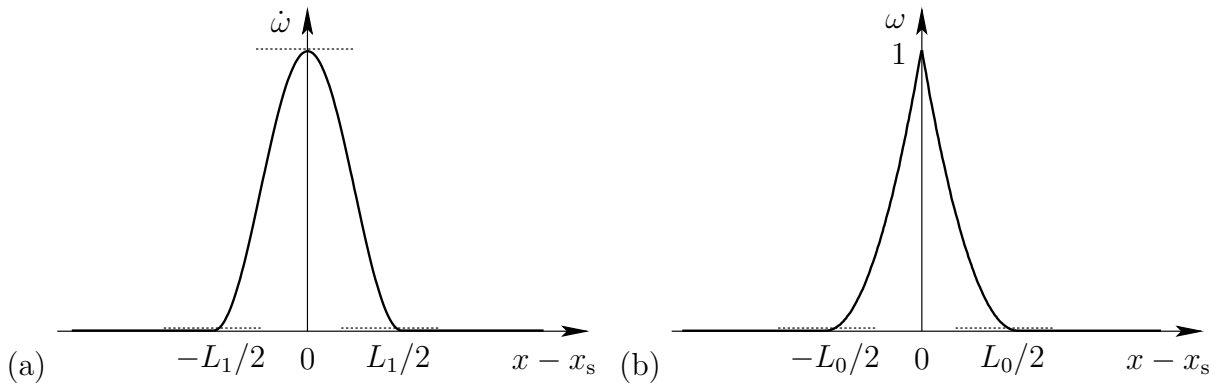


FIGURE 2. Sketch of bifurcation shapes; (a) initial ($\mu = 1$), (b) final ($\mu = 0$).

3.2. Complete damage. To gain insight into the behavior at complete failure, observe that Eq. (13), formally written for $\mu = 0$, leads to a differential equation

$$\ell^2 \omega''(x) = 1 \quad (21)$$

which admits solutions in the form

$$\omega(x) = \frac{(x - x_s)^2}{2\ell^2} + C_2(x - x_s) + C_3 \quad (22)$$

Analysis analogous to the treatment of damage initiation leads to the integration constants

$$L_0 = 2\sqrt{2}\ell, \quad C_2 = \frac{\sqrt{2}}{\ell^3}, \quad C_3 = 1 \quad (23)$$

resulting in the bifurcation shape shown in Figure 2b. Clearly, such a solution is not compatible with the irreversibility constraint, since $L_0 < L_1$. Nevertheless, it reveals the tendency of the damaging zone to shrink as μ decreases from 1 to 0.

3.3. Numerical solution. The previous analysis demonstrates that the irreversibility condition plays an important role in the damage evolution. Therefore, the damage profile evolution needs to be determined by integrating the rate form Eq. (15) numerically, supplemented with irreversibility condition $\dot{\omega} > 0$. As the goal of the current paper is to illustrate the basic feature of the model rather than to develop an efficient numerical scheme, the resulting problem is resolved by the implicit Euler (or Rothe-Rektorys) scheme with the damage rate $\dot{\omega}$ determined using the `bvp4c` solver (Kierzenka and Shampine, 2001) implemented in MATLAB[®]. The corresponding damage profile evolution, appearing in Figure 3a, corresponds to $N = 200$ time steps and spatial discretization of the interval $(x_s, x_s + L_1/2)$ into 200 elements. Moreover, the associated load-displacement diagram appears in Figure 3b to demonstrate that the model can properly describe complete failure. Both results confirm a physically admissible response of the studied model, free of locking effects.

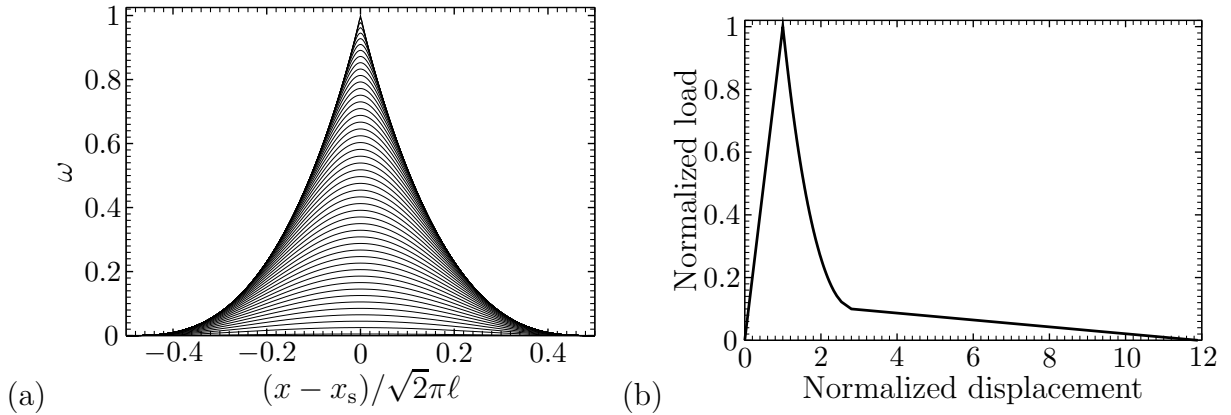


FIGURE 3. Localization study; (a) evolution of damage profile, (b) load-displacement diagram.

Acknowledgments. The authors would like to acknowledge the financial support from the Ministry of Education, Youth and Sport of the Czech Republic under Research Plan MSM 6840770003.

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